Section 4.2

Maximum and Minimum Values

(1) Absolute and Local Extrema

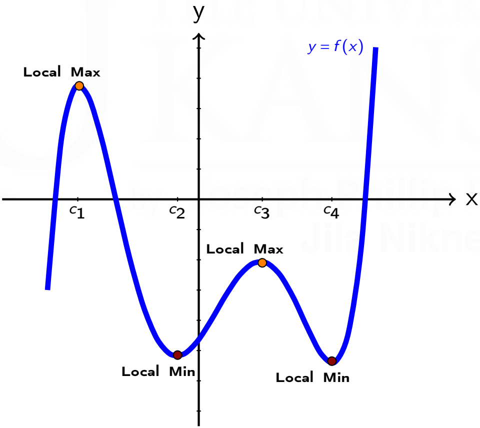
(2) The Extreme Value Theorem

(3) The Closed Interval Method

Local Extrema

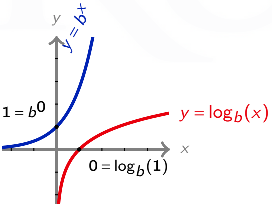
A function f has a **local maximum** at c if $f(c) \geq f(x)$ for x "near" c. That is, $f(c) \geq f(x)$ for all x in some open interval containing c.

A function f has a **local minimum** at c if $f(c) \leq f(x)$ for x "near" c.



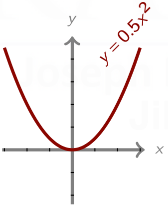
Local Extrema

- The term “extremum” is shorthand for “maximum or minimum.”
- If f has a local extremum at c, then $y = f(c)$ is a **local extreme value** and $(c, f(c))$ is a **local extreme point**.
- An endpoint of the domain f of cannot be a local extremum, because it cannot be contained in any open interval in the domain.
- A function does not necessarily have to have any local extrema:



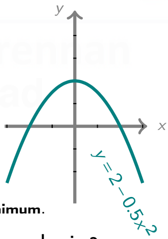
No maximum.

No minimum.



No maximum.

Minimum value is 0.



No minimum.

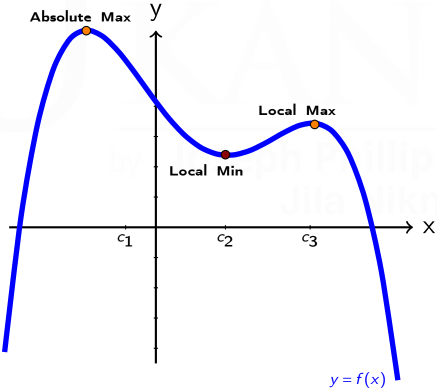
Maximum value is 2.

Absolute Extrema

A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for **all** x.

A function f has an **absolute minimum** at c if $f(c) \leq f(x)$ for **all** x.

- Unless it is an endpoint, each absolute extremum is also a local extremum.



Example 1: Absolute Extrema

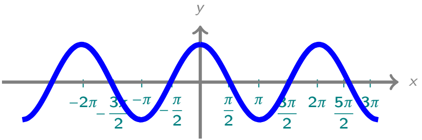
A function can have **at most one** absolute maximum value, but **any** number of absolute maximum points.

Example 1(a): $f(x) = x^2$ has **no absolute maximum**, because x^2 can be arbitrarily large.

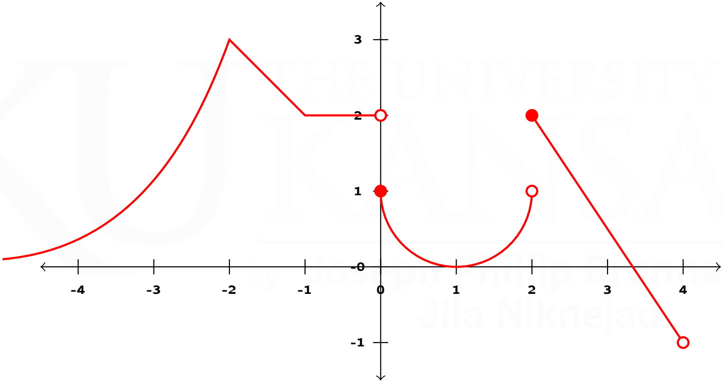
Example 1(b): $f(x) = -x^2$ has **absolute maximum** value 0 at the point $(0,0)$.

Example 1(c): $f(x) = \cos(x)$ has absolute maximum value 1 and infinitely many absolute maximum points: $(k\pi, 1)$ where k is any even integer.

$y = \cos(x)$



Example 2: Local and Absolute Maxima and Minima



Local maxima: $x = -2$, $x = 2$, all x in $(-1, 0)$
Local minima: $x = 1$, all x in $[-1, 0)$
Absolute maximum: $(-2, 3)$ **Absolute minimum:** None

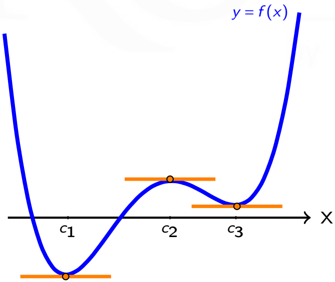
Critical Numbers and Fermat's Theorem

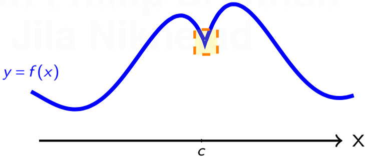
A number c in the domain of f is called a critical number if either $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem

If f has a local extremum at $x = c$, and $f'(c)$ exists, then $f'(c) = 0$.

That is, if f has a local max or min at c, then c is a critical number of f.





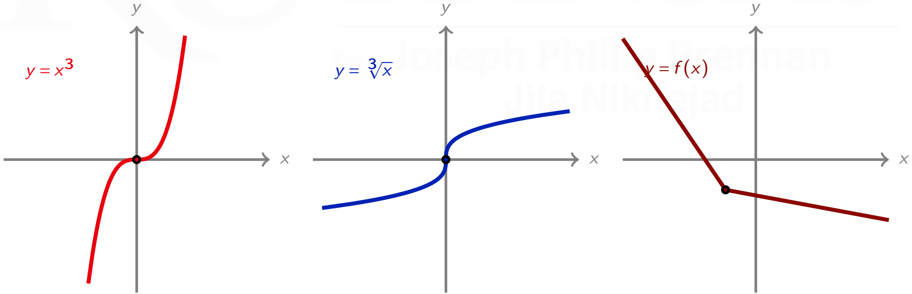
Critical Numbers

Fermat's Theorem

If f has a local extremum at $x = c$, and $f'(c)$ exists, then $f'(c) = 0$.

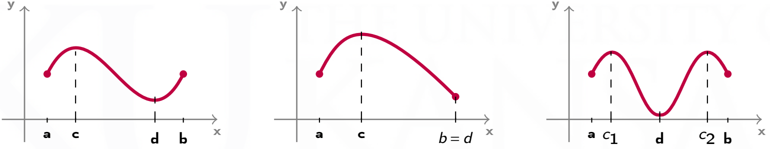
That is, if f has a local max or min at c, then c is a critical number of f.

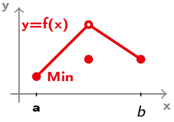
On the other hand, not every critical point must be a local max or min.



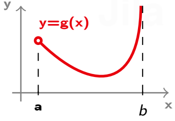
The Extreme Value Theorem

If f is continuous on a **closed** interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

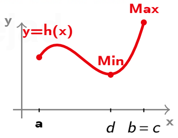




(A) A discontinuous function on interval $[a, b]$ has a min but no max on interval $[a, b]$.



(B) A continuous function on interval (a, b) has no min or max on open interval (a, b).



(C) Every continuous function on a closed interval $[a, b]$ has both a min and a max on $[a, b]$.

The Closed Interval Method

The Extreme Value Theorem

If f is continuous on a **closed** interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

How do we systematically find the absolute extrema?

The Closed Interval Method

To find the absolute extreme values of a continuous function f on a closed interval $[a, b]$:

(1) Find the values of f at the **critical numbers** in (a, b).

(2) Find the values of f at the **endpoints** (namely a and b).

(3) Compare the y-values. The largest value is the absolute maximum value; the smallest value is the absolute minimum value.

The Closed Interval Method

Example 3: Find the absolute extrema of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on the closed interval $[0,6]$.

Note that f is a polynomial, so it is continuous everywhere.

1. Check the critical numbers.

$f'(x) = 6x^2 - 30x + 24 = 6(x - \underline{1})(x - \underline{4})$.

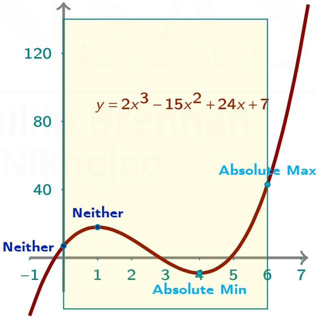
$f(1) = 18$ and $f(4) = -9$.

2. Check the endpoints.

$f(0) = 7$ and $f(6) = 43$.

3. Compare the y-values.

Absolute minimum: $(4, -9)$
Absolute maximum: $(6, 43)$



Finding Extrema

Example 4: Find the absolute extrema of $h(x) = x^{4/5}(x-4)^2$ on $[1,5]$.

Solution: The function h is continuous on $[1,5]$, so the Extreme Value Theorem guarantees that it has absolute extrema and we can use the Closed Interval Method.

$$h'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + 2x^{4/5}(x-4) = \frac{4(x-4)^2 + 10x(x-4)}{5x^{1/5}}$$

$$h'(x) = \frac{2(x-4)(7x-8)}{5x^{\frac{1}{5}}}$$

- Critical numbers: $0, \frac{8}{7}, 4$. We can ignore 0 because it is outside $[1,5]$.
- Critical points and endpoints (y-coordinates approximate):

$$(1,9) \qquad (8/7, \mathbf{9.08}) \qquad (4, \mathbf{0}) \qquad (5, 3.62)$$

Finding Extrema

Example 4: Find the absolute extrema of $h(x) = x^{4/5}(x-4)^2$ on $[1,5]$.